

THRESHOLD EFFECTS IN RELATIVISTIC GASES

V.V. Begun,¹ L. Ferroni,² M.I. Gorenstein,^{1,3} M. Gaździcki,^{4,5} and F. Becattini²

¹*Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine*

²*Università di Firenze and INFN Sezione di Firenze*

³*Frankfurt Institute for Advanced Studies, Frankfurt, Germany*

⁴*Institut für Kernphysik, Johann Wolfgang Goethe Universität Frankfurt, Germany*

⁵*Świętokrzyska Academy, Kielce, Poland*

Abstract

Particle multiplicities and ratios in the microcanonical ensemble of relativistic gases near production thresholds are studied. It is shown that the ratio of heavy to light particle multiplicity may be enhanced in comparison to its thermodynamic limit.

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I. INTRODUCTION

The statistical model of hadron production in high energy collisions of both elementary and heavy ion collisions proved to be remarkably successful in reproducing the multiplicities of different hadron species. This finding has raised the question of the meaning of thermodynamical quantities in these systems and triggered a still ongoing debate [1, 2].

In the case of relativistic heavy ion collisions, due the large multiplicities involved, the use of the grand canonical ensemble (GCE) is sufficient and the model predicts that hadron yields should be linearly dependent on the volume at hadronization. On the other hand, in elementary collisions, the canonical ensemble (CE) [1, 3, 4, 5, 6] enforcing the exact conservation of charges must be used owing to the relatively low multiplicities. At lower multiplicities, even the micro-canonical ensemble (MCE), where energy-momentum conservation is enforced, is needed to correctly describe particle abundances. The transition from MCE to CE has been studied in detail in refs. [7, 8]. The equivalence between the three ensembles, with respect to particle multiplicities, is recovered only in the thermodynamic limit, when $V \rightarrow \infty$.

The charged conservation laws in the CE imply a reduction of the mean multiplicity of heavy charged hadrons in comparison to the corresponding multiplicity calculated in the GCE. This effect is usually defined as *canonical suppression* and was particularly studied for the production of strange [4] and open charm hadrons [6] as well as anti-baryons [5].

The canonical suppression is significant for the charged particles with a mean multiplicity smaller than 1. Naively one may expect that an introduction of the energy conservation law by use of the MCE should lead to the gradually increasing suppression of the multiplicity of heavy hadrons with decreasing energy of the system. Contrary to this expectation, in this work we demonstrate that the energy and momentum conservation laws lead to rich structures in the dependence of hadron multiplicities on the system energy.

The paper is organized as follows: in Section II we study this effect by means of a simple system of massless and heavy particles, where an analytical description is possible. In Sect. III, we study the full ideal hadron-resonance gas by means of numerical methods described in refs. [7, 8]. Conclusions are summarized in Sect. IV.

II. THE ANALYTICAL MODEL

In this section we consider the system of noninteracting Boltzmann particles with zero mass, both neutral (with degeneracy factor g) and charged (with degeneracy factor g_{\pm}). The system also includes neutral heavy particles with mass M and degeneracy factor G . The nonrelativistic approximation is used for heavy particles.

A. Grand Canonical Ensemble

In the GCE independent variables are volume V , temperature T , and chemical potential μ . The average number of massless neutral particles N , massless charged particles, N_+ and N_- , and heavy neutral particles, n , are:

$$\begin{aligned}\langle N \rangle_{GCE} &= \frac{gV T^3}{\pi^2}, \quad \langle N_{\pm} \rangle_{GCE} = \frac{g_{\pm} V T^3}{\pi^2} \exp\left(\pm \frac{\mu}{T}\right), \\ \langle n \rangle_{GCE} &= GV \left(\frac{MT}{2\pi}\right)^{3/2} \exp\left(-\frac{M}{T}\right).\end{aligned}\tag{1}$$

The system energy reads:

$$\langle E \rangle_{GCE} \equiv \varepsilon(T) V = 3T \langle N \rangle_{GCE} + 3T \langle N_+ \rangle_{GCE} + 3T \langle N_- \rangle_{GCE} + \left(\frac{3}{2}T + M\right) \langle n \rangle_{GCE}.\tag{2}$$

Of course, the nonrelativistic approximation used for massive particles holds for $M \gg T$.

B. Micro-Canonical Ensemble

In the MCE independent variables are volume V , energy E , and net charge Q . The MCE partition function $\Omega_N(E, V)$ in the system of N neutral massless particles equals to [9] :

$$\Omega_N(E, V) = \frac{1}{N!} \left(\frac{gV}{2\pi^2}\right)^N \int_0^\infty p_1^2 dp_1 \dots \int_0^\infty p_N^2 dp_N \delta\left(E - \sum_{j=1}^N p_j\right) = \frac{1}{N!} \left(\frac{gV}{\pi^2}\right)^N \frac{E^{3N-1}}{(3N-1)!}.\tag{3}$$

Let us consider the case of charged massless particles with total zero net charge, $Q = 0$. This implies $\mu_Q = 0$ in the GCE and thus $\langle N_+ \rangle_{GCE} = \langle N_- \rangle_{GCE}$. The MCE partition function for $N_+ = N_-$ charged massless particles reads [9]:

$$\Omega_{N_{\pm}}(E, V) = \frac{1}{N_{\pm}!^2} \left(\frac{g_{\pm} V}{\pi^2}\right)^{2N_{\pm}} \frac{E^{6N_{\pm}-1}}{(6N_{\pm}-1)!},\tag{4}$$

where $N_{\pm} \equiv N_+ = N_-$.

The MCE partition function for n heavy nonrelativistic particles can be calculated analytically:

$$\begin{aligned}\Omega_n(E, V) &\equiv \frac{1}{n!} \left(\frac{GV}{2\pi^2} \right) \int_0^\infty p_1^2 dp_1 \dots \int_0^\infty p_n^2 dp_n \delta \left[E - \sum_{j=1}^n \left(M + \frac{p_j^2}{2M} \right) \right] \\ &= \frac{1}{n!} \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \frac{M^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2})} (E - nM)^{\frac{3n}{2}-1},\end{aligned}\quad (5)$$

where the Euler gamma function $\Gamma(x)$ has a simple form for integer k and halfinteger $k + 1/2$ arguments:

$$\Gamma(k) = (k-1)!, \quad \Gamma\left(k + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2^n} \sqrt{\pi}.$$

Using Eqs. (3, 5) one can calculate the partition function for N massless and n heavy particles:

$$\begin{aligned}\Omega_{N,n}(E, V) &= \int_0^\infty dE_1 \int_0^\infty dE_2 \Omega_N(E_1, V) \Omega_n(E_2, V) \delta[E - E_1 - E_2] \\ &= \frac{1}{N!} \frac{1}{n!} \left(\frac{gV}{\pi^2} \right)^N \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \frac{M^{\frac{3n}{2}}}{\Gamma(3N + \frac{3}{2}n)} (E - nM)^{3N + \frac{3}{2}n - 1}.\end{aligned}\quad (6)$$

Note that the MCE partition functions $\Omega_N(E, V)$ (3) and $\Omega_n(E, V)$ (5) are defined for nonzero particle numbers, $N \geq 1$ and $n \geq 1$, because of an exact energy conservation. However, the MCE partition functions $\Omega_{N,n}(E, V)$ (6) requires only $N + n \geq 1$, so that either N or n can be equal to zero.

Finally, using Eqs. (4, 6) one finds the partition function for N massless neutral, $N_+ = N_-$ massless charged, and n heavy particles:

$$\begin{aligned}\Omega_{N,N_{\pm},n}(E, V) &= \int_0^\infty dE_1 \int_0^\infty dE_2 \Omega_{N,n}(E_1, V) \Omega_{N_{\pm}}(E_2, V) \delta[E - E_1 - E_2] \\ &= \frac{1}{N!} \frac{1}{(N_{\pm})!^2} \frac{1}{n!} \left(\frac{gV}{\pi^2} \right)^N \left(\frac{g_{\pm}V}{\pi^2} \right)^{2N_{\pm}} \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \times \\ &\quad \times \frac{M^{\frac{3n}{2}}}{\Gamma(3N + 6N_{\pm} + \frac{3}{2}n)} (E - nM)^{3N + 6N_{\pm} + \frac{3}{2}n - 1}.\end{aligned}\quad (7)$$

The MCE partition function $\Omega_{N,N_{\pm},n}(E, V)$ (7) is evidently transformed into $\Omega_{N,n}(E, V)$ (6) for $g_{\pm} = 0$. For $g = 0$ the $\Omega_{N,N_{\pm},n}(E, V)$ from Eq. (7) is transformed into the partition function for $N_+ = N_-$ massless charged and n heavy particles:

$$\Omega_{N_{\pm},n}(E, V) = \frac{1}{(N_{\pm})!^2} \frac{1}{n!} \left(\frac{g_{\pm}V}{\pi^2} \right)^{2N_{\pm}} \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \frac{M^{\frac{3n}{2}}}{\Gamma(6N_{\pm} + \frac{3}{2}n)} (E - nM)^{6N_{\pm} + \frac{3}{2}n - 1}. \quad (8)$$

The MCE partition function can also be calculated for the case of non-zero system net charge Q in two steps:

$$\Omega_{N_+, N_-}(E, V) = \int_0^\infty dE_1 \int_0^\infty dE_2 \Omega_{N_+}(E_1, V) \Omega_{N_-}(E_2, V) \delta[N_+ - N_- - Q] \delta[E - E_1 - E_2], \quad (9)$$

$$\Omega_{N_+, N_-, n}(E, V) = \int_0^\infty dE_1 \int_0^\infty dE_2 \Omega_{N_+, N_-}(E_1, V) \Omega_n(E_2, V) \delta[E - E_1 - E_2]. \quad (10)$$

Finally, the partition function for N_+ positively charged massless particles and for n neutral heavy particles equals to:

$$\begin{aligned} \Omega_{N_+, n}(E, V, Q) &= \frac{1}{N_+!} \frac{1}{(N_+ - Q)!} \frac{1}{n!} \left(\frac{g_\pm V}{\pi^2} \right)^{2N_+ - Q} \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \times \\ &\times \frac{M^{\frac{3n}{2}}}{\Gamma(6N_+ - 3Q + \frac{3}{2}n)} (E - nM)^{6N_+ - 3Q + \frac{3}{2}n - 1}, \end{aligned} \quad (11)$$

and for negatively charged massless particles:

$$\begin{aligned} \Omega_{N_-, n}(E, V, Q) &= \frac{1}{N_-!} \frac{1}{(N_- + Q)!} \frac{1}{n!} \left(\frac{g_\pm V}{\pi^2} \right)^{2N_- + Q} \left(\frac{GV}{(2\pi)^{3/2}} \right)^n \times \\ &\times \frac{M^{\frac{3n}{2}}}{\Gamma(6N_- + 3Q + \frac{3}{2}n)} (E - nM)^{6N_- + 3Q + \frac{3}{2}n - 1}. \end{aligned} \quad (12)$$

These formulas transform into $\Omega_{N_\pm, n}(E, V)$ (8) for $Q = 0$.

Now we calculate the mean particle multiplicities in the MCE and compare them to the GCE multiplicities (1). This comparison is performed at the same energy density which is set to be the energy density calculated in the GCE for $T = 160$ MeV. In the MCE the volume V equals to that in the GCE, and the MCE energy E equals to GCE average energy E_{GCE} (2). Let us start with the system of massless charged and heavy neutral particles with zero net charge. In this case all MCE averages are calculated as:

$$\langle \dots \rangle_{MCE} = \frac{1}{\Omega(E, V; Q = 0, M)} \sum_{N_\pm, n=0}^\infty \dots \Omega_{N_\pm, n}(E, V), \quad (13)$$

where $\Omega_{N_\pm, n}(E, V)$ is given by Eq. (8), $\Omega(E, V; Q = 0, M) \equiv \sum_{N_\pm, n=0}^\infty \Omega_{N_\pm, n}(E, V)$, and $\Omega_{0,0}(E, V) = 0$, $n_{max} = \lceil \frac{E}{M} \rceil$ due to the exact energy conservation. The degeneracy factors are fixed to be $g_\pm = G = 1$. In Fig. 1 the ratios of average particle numbers in the MCE to those in the GCE are presented as functions of the system energy $E \equiv \langle E \rangle_{GCE}$. We remind

that $V \equiv E/\varepsilon(T)$. Circles and triangles represent Monte-Carlo (MC) calculations for the same parameters (see Section III for details of the used procedure). These calculations are in an agreement with the analytical results. Further we also show the MC results obtained with the additional requirement of the exact momentum conservation ($\mathbf{P} = \mathbf{0}$) for the whole system. For the heavy particles with mass $M = 3.1$ GeV one observes (see Fig. 1) the MCE suppression of the heavy particle multiplicity. This suppression is strong at energies E comparable to heavy particle mass M , and it is still significant at $E = 100$ GeV $\gg M$.

In Fig. 2 (left) the ratios $\langle N_{\pm} \rangle_{MCE} / \langle N_{\pm} \rangle_{GCE}$ and $\langle n \rangle_{MCE} / \langle n \rangle_{GCE}$ are shown for a smaller value of the heavy particle mass, $M = 0.7$ GeV. One observes a fast drop of the $\langle N_{\pm} \rangle_{MCE} / \langle N_{\pm} \rangle_{GCE}$ and strong “oscillations” of the $\langle n \rangle_{MCE} / \langle n \rangle_{GCE}$ ratio. The MCE enhancement effect ($\langle n \rangle_{MCE} / \langle n \rangle_{GCE} > 1$) increases strongly with decreasing mass M of the heavy particle. Note that the non-relativistic treatment of the heavy particles is still approximately valid at $M = 0.7$ GeV and $T = 160$ MeV.

Let us consider the massless charged and heavy neutral particles with the net charge $Q = 2$. In this case all MCE averages are also calculated by means of Eq. (13). Besides, $\langle N_+ \rangle_{MCE}$ and $\langle N_- \rangle_{MCE}$ are calculated independently using Eqs. (11,12). The corresponding GCE values now contain $\exp(\pm\mu/T)$, and the equation $\langle N_+ \rangle_{GCE} - \langle N_- \rangle_{GCE} = Q$ should be solved:

$$\frac{g_{\pm} V T^3}{\pi^2} \left[\exp\left(\frac{\mu}{T}\right) - \exp\left(-\frac{\mu}{T}\right) \right] = Q, \quad (14)$$

where the volume is given by Eq. (2) as $\langle E \rangle_{GCE} / \varepsilon(T)$. In order to make a simple estimate, we neglect a small contribution of heavy particles to the energy density and consequently Eq. (14) transforms into:

$$\frac{E}{3T} \tanh\left(\frac{\mu}{T}\right) = Q. \quad (15)$$

This equation has solutions for $E > 3TQ$. The results for the MCE to GCE particle ratios are presented in Fig. 2 (right). Negatively charged particles are suppressed because of the exact charge conservation, and the first maximum seen in Fig. 2 (left) for heavy neutral particles disappears.

C. Canonical Ensemble

Let us consider the CE system with zero total net charge, $Q = 0$. A standard way is to compare the CE and GCE results at the same volume V and temperature T (see e.g. [4, 5, 6]). One observes here the reduction of charged particle multiplicities with decreasing system volume in comparison to ones calculated in the thermodynamical limit. The effect is called canonical suppression. Note that the energy density in the CE, ε_{CE} , is also suppressed in comparison with that in the GCE at the same temperature. This is due to the suppression of charged particle multiplicities. Our aim is to compare the particle multiplicities in different ensembles at the same energy density. Thus, we perform the comparison at the same volume and energy in all ensembles, so that the energy densities are also equal to each other, $\varepsilon_{MCE} = \varepsilon_{GCE} = E/V = \text{const}$. Of course, under this condition the mean multiplicities and temperatures calculated in the CE and the GCE are equal in the thermodynamic limit, $V \rightarrow \infty$. The results are, however, different for small systems. When the volume of the system in the CE decreases, but the energy density is kept fixed, the temperature increases which leads to an increase of the heavy neutral particle multiplicity. The CE temperature, T^* , is obtained as a solution of the equation:

$$\frac{\langle E \rangle_{CE}}{V} \equiv 6T^* \frac{\langle N_{\pm} \rangle_{CE}}{V} + \left(\frac{3}{2}T^* + M \right) \frac{\langle n \rangle_{CE}}{V} = \frac{E}{V} = \varepsilon_{GCE} = \text{const}, \quad (16)$$

where ($z \equiv gVT^{*3}/\pi^2$)

$$\langle N_{\pm} \rangle_{CE} = z \frac{I_1(2z)}{I_0(2z)}, \quad \langle n \rangle_{CE} = GV \left(\frac{MT^*}{2\pi} \right)^{3/2} \exp \left(-\frac{M}{T^*} \right). \quad (17)$$

A comparison of Eq. (17) and Eq. (1) demonstrates the CE suppression of charged multiplicities. This leads to $T^* > T$ due to Eq. (16) hence to $\langle n \rangle_{CE} > \langle n \rangle_{GCE}$, as the equations for neutral particle multiplicities have the same form in the GCE and the CE. The results are shown in Fig. 3. Massless particles are suppressed in the CE, Fig. 3 (left), since the increase of temperature is insufficient to overcome the CE suppression. A smooth increase of the ratio $\langle n \rangle_{CE}/\langle n \rangle_{GCE}$ is seen in Fig. 3 (right). Thus, a fast increase just above threshold and the following “oscillations” of the $\langle n \rangle_{MCE}/\langle n \rangle_{GCE}$ ratio are due to energy-momentum conservation laws.

D. Grand Micro-Canonical Ensemble

We turn now to the discussion of the system with neutral particles only, both massless and heavy ones. When the energy is exactly conserved we call this ensemble the grand micro-canonical ensemble (GMCE) in order to distinguish it from the MCE where both the energy and the charge are strictly conserved. The calculation within the GMCE allows us to check whether the MCE enhancement is only related to the charge conservation or it is due to the energy conservation. We set $g_{\pm} = 0$ and $g = G = 1$, and in this case all GMCE averages are calculated as:

$$\langle \dots \rangle_{GMCE} = \frac{1}{\Omega(E, V; M)} \sum_{N, n=0}^{\infty} \dots \Omega_{n, N}(E, V), \quad (18)$$

where $\Omega(E, V; M) \equiv \sum_{N, n=0}^{\infty} \Omega_{n, N}(E, V)$ is the total MCE partition function, where $\Omega_{0,0}(E, V) = 0$ and $n_{max} = \left\lfloor \frac{E}{M} \right\rfloor$ due to the exact energy conservation. The result is shown in Fig. 4, where one can see that the heavy particle enhancement near the threshold is also present in the neutral system. The amplitude of a drop in $\langle N \rangle_{GMCE} / \langle N \rangle_{GCE}$ and the “oscillations” of $\langle n \rangle_{GMCE} / \langle n \rangle_{GCE}$ becomes smaller than in the MCE with charged massless particles, but they are still present. The Monte-Carlo (MC) calculations with the total momentum conservation show a similar behavior: a decrease of the ratio for the massless particles and an increase for the heavy particles. The momentum conservation leads to an increase of both ratios, it also smears and shifts the peak for heavy particles. In Fig. 5 we present the energy dependence of the particle multiplicities calculated in the GMCE and the GCE.

III. THE HADRON-RESONANCE GAS MODEL

This section starts with a brief presentation of the numerical methods used to calculate mean hadron multiplicities in the MCE for the ideal hadron-resonance gas. The full description can be found in [7, 8]. Further on, we show and discuss the results on the energy dependence of hadron yields near the threshold.

A. The Procedure

The MCE partition function is defined as the sum over all multi-hadronic states $|h_V\rangle$ localized within the volume V of the system and constrained with the four-momentum and the abelian (i.e. additive) charge conservation:

$$\Omega = \sum_{h_V} \langle h_V | \delta^4(P - P_{\text{op}}) \delta_{\mathbf{Q}, \mathbf{Q}_{\text{op}}} | h_V \rangle, \quad (19)$$

where $\mathbf{Q} = (Q_1, \dots, Q_M)$ is a vector of M integer abelian charges (electric, baryon number, strangeness etc.), P is the overall four-momentum of the system and $P_{\text{op}}, \mathbf{Q}_{\text{op}}$ the relevant operators. Provided that relativistic quantum field effects are neglected and the volume of the system is large enough to allow the approximation of finite-volume Fourier integrals with Dirac deltas, it can be proved [7] that the micro-canonical partition function Ω can be written as a multiple integral:

$$\Omega = \frac{1}{(2\pi)^{4+M}} \int d^4y e^{iP \cdot y} \int_{-\pi}^{+\pi} d^M \phi e^{i\mathbf{Q} \cdot \phi} \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3p \log(1 \pm e^{-ip_j \cdot y - i\mathbf{q}_j \cdot \phi})^{\pm 1} \right] \quad (20)$$

where \mathbf{q}_j is the vector of the abelian charges for the j^{th} hadron species, J_j its spin and $p_j = (\sqrt{m_j^2 + \mathbf{p}^2}, \mathbf{p})$, where m_j is the mass of the j^{th} hadron species; the upper sign applies to fermions, the lower to bosons. The integral (20) is more easily calculable in the rest frame of the system where $\mathbf{P} = \mathbf{0}$. Unfortunately, an analytical solution with no charge constraint is known only in two limiting cases treated in previous sections: non-relativistic and ultra-relativistic (i.e. with all particle masses set to zero). The full relativistic case has been attacked with several kinds of expansions [10] but none of them proved to be fully satisfactory as the achieved accuracy in the estimation of different kinds of averages could vary from some percent to a factor 10. Therefore, a numerical integration of Eq. (20) is needed. The most suitable method is to decompose Ω into the sum of the phase space volumes with fixed particle multiplicities for each species:

$$\Omega = \sum_{\{N_j\}} \Omega_{\{N_j\}} \delta_{\mathbf{Q}, \sum_j N_j \mathbf{q}_j}, \quad (21)$$

$\{N_j\}$ being a vector of K integer numbers (N_1, \dots, N_K) , i.e. the multiplicities of all of the K hadronic species.

Here we use the approximated expression in Boltzmann statistic of the phase space volume $\Omega_{\{N_j\}}$ for the channel $\{N_j\}$ in order to make a qualitative study on micro-canonical effects in a

coherent way with the analysis in the analytical model made in the previous part of the paper. For the same reason, we will keep fixed masses for hadron resonances neglecting the Breit Wigner broadening effect. Both of these simplifications, do not affect the results significantly. The phase space volume for fixed multiplicities in Boltzmann statistic reads [7]:

$$\Omega_{\{N_j\}} = \prod_j \frac{V^{N_j} (2J_j + 1)^{N_j}}{(2\pi)^{3N_j} N_j!} \int d^3p_1 \dots d^3p_N \delta^4(P - \sum_{i=1}^N p_i), \quad (22)$$

where $N = \sum_j N_j$.

We are mainly interested in the calculation of quantities relevant to particle multiplicities, not to their momenta, namely their kinematical state. The average of an observable O depending on particle multiplicities in the micro-canonical ensemble can then be written as:

$$\langle O \rangle = \frac{\sum_{\{N_j\}} O(\{N_j\}) \Omega_{\{N_j\}} \delta_{\mathbf{Q}, \sum_j N_j \mathbf{q}_j}}{\sum_{\{N_j\}} \Omega_{\{N_j\}} \delta_{\mathbf{Q}, \sum_j N_j \mathbf{q}_j}}, \quad (23)$$

where $O(\{N_j\}) \equiv N_k$ for the average multiplicity of the k -th hadron species $\langle N_k \rangle$. Altogether, what we need to calculate in order to evaluate an average (23) are integrals like (22).

In order to be able to effectively calculate $\Omega_{\{N_j\}}$ for any channel, a brute force option is to do it for all of them. However, this method is not appropriate for a system like the hadron gas, as the actual number of channels is very large. Indeed, with 265 light-flavored hadrons and resonances (those included in the latest Particle Data Book issue [11]), the number of channels allowed by the energy-momentum conservation is enormous and increases almost exponentially with the cluster mass, which requires an unacceptably large computing time. Therefore, the calculation of the phase space volume of all allowed channels is only possible for very light clusters, in practice lighter than ~ 2 GeV. Hence, if a method based on the exhaustive exploration of the channel space is not affordable, one has to resort to Monte-Carlo methods, whereby the channel space is randomly sampled.

An estimate of the average (23) can be made by means of the so-called importance sampling method. The idea of this method is to sample the channel space (i.e. the set of integers N_j , one for each hadron species) not uniformly, but according to an auxiliary distribution $\Pi_{\{N_j\}}$ which must be suitable to being sampled very efficiently to keep computing time low, and, at the same time, as similar as possible to the distribution $\Omega_{\{N_j\}}$. The latter requirement is dictated by the fact that $\Omega_{\{N_j\}}$ is sizable over a very small portion of the whole channel space. Thus, if random configurations were generated uniformly, for almost all of them $\Omega_{\{N_j\}}$ would

have a negligible value, and a huge number of samples would be required to achieve a good accuracy. On the other hand, if samples are drawn according to a distribution similar to $\Omega_{\{N_j\}}$, little time is wasted to explore unimportant regions and the estimation of the average (23) is more accurate. A crucial requirement for $\Pi_{\{N_j\}}$ is not to be vanishing or far smaller than $\Omega_{\{N_j\}}$ anywhere in its domain in order not to exclude some good regions from being sampled, thereby biasing the calculated averages in a finite statistics calculation.

A Monte-Carlo estimate of $\langle O \rangle$ is:

$$\langle O \rangle \doteq \frac{\sum_{k=1}^{N_S} O(\{N_j\}^{(k)}) \frac{\Omega_{\{N_j\}}^{(k)}}{\Pi_{\{N_j\}}^{(k)}}}{\sum_{k=1}^{N_S} \frac{\Omega_{\{N_j\}}^{(k)}}{\Pi_{\{N_j\}}^{(k)}}}, \quad (24)$$

where $\{N_j\}^{(k)}$ are samples of the channel space extracted according to the distribution Π and fulfilling the charge constraint $\mathbf{Q} = \sum_j N_j \mathbf{q}_j$.

Provided that N_S is large enough so that the distributions of both numerator and denominator in Eq. (24) are Gaussians (hence the conditions of validity of the central limit theorem are met), the statistical error $\sigma_{\langle O \rangle}$ on the average $\langle O \rangle$ can be estimated as:

$$\sigma_{\langle O \rangle}^2 = \frac{1}{N_S \Omega^2} \left\{ \mathbb{E}_{\Pi} \left(O^2 \frac{\Omega_{\{N_j\}}^2}{\Pi_{\{N_j\}}^2} \right) + \langle O \rangle^2 \mathbb{E}_{\Pi} \left(\frac{\Omega_{\{N_j\}}^2}{\Pi_{\{N_j\}}^2} \right) - 2 \langle O \rangle \mathbb{E}_{\Pi} \left(O \frac{\Omega_{\{N_j\}}^2}{\Pi_{\{N_j\}}^2} \right) \right\}, \quad (25)$$

where \mathbb{E}_{Π} stands for the expectation value relevant to the Π distribution.

Here we define $\Pi_{\{N_j\}}$ as the product of K (as many as particle species) Poisson distributions:

$$\Pi_{\{N_j\}} = \prod_{j=1}^K \exp[-\nu_j] \frac{\nu_j^{N_j}}{N_j!} \quad (26)$$

which will be henceforth referred to as the *multi-Poisson distribution* or MPD, enforcing as mean values the mean hadronic multiplicities ν_j calculated in the GCE with volume and mean energy equal to the volume and mass of the system:

$$\nu_j = \frac{(2J_j + 1)V}{2\pi^2} m_j^2 T K_2 \left(\frac{m_j}{T} \right) \prod_i \lambda_i^{q_{ji}}, \quad (27)$$

where V is the volume of the system, T is the temperature and λ_i the fugacity corresponding to the charge Q_i . Temperature and fugacities are determined by enforcing the GCE mean energy

and charges to be equal to the actual energy E and charges \mathbf{Q} of the system:

$$\begin{aligned} E &= T^2 \frac{\partial}{\partial T} \sum_j z_j(T) \prod_i \lambda_i^{q_{ji}} , \\ \mathbf{Q} &= \sum_j \mathbf{q}_j z_j(T) \prod_i \lambda_i^{q_{ji}} , \end{aligned} \quad (28)$$

with

$$z_j(T) = \frac{(2J_j + 1)V}{2\pi^2} m_j^2 T K_2 \left(\frac{m_j}{T} \right) . \quad (29)$$

The distribution (26) can indeed be sampled very efficiently and is the actual multi-species multiplicity distribution in the GCE in the limit of Boltzmann statistics.

B. Threshold Effects in the Hadron-Resonance Gas

We have calculated the ratios of mean multiplicities of several hadron species in the MCE and the CE with respect to the GCE for an ideal hadron-resonance gas including all species up to a mass of about 1.8 GeV. Quantum statistic effects and resonance mass broadening have been turned off. In order to make a proper comparison between the different ensembles, energy (in the rest frame) and volume in both the CE and the GCE have been set to the same value as in the MCE. This implies that the energy density ε has the same value in all ensembles as well. It has been fixed to 0.3895 GeV/fm³, corresponding to a temperature of 160 MeV in the GCE. As it has been discussed in Section II, in the CE, owing to the exact charge conservation, the energy density is not an intensive quantity but it also depends on the volume. This implies that for a fixed energy density, temperature varies as a function of total energy. In particular, for a completely neutral hadron gas which will be discussed henceforth temperature decreases as energy increases. This happens because at lower energies or volumes charge conservation suppresses more and more charged particle multiplicities and the system needs an increase in temperature to keep the energy density constant.

The results of our calculations are shown in Figs. 6-13, as a function of the total energy near the production threshold. In general, it can be seen from these plots that for an actual ideal hadron-resonance gas the (micro-)canonical enhancement for some particle species near the threshold (which was implicitly numerically observed in [8]) shows the same qualitative features as those of the analytical model. Nevertheless, the correlation effects between different particle species due to charge and the energy-momentum conservation play an important role

from the quantitative point of view and results in abrupt changes in the micro-canonical mean multiplicities as the total energy exceeds the threshold for the production of a new channel.

In Fig. 6 we present the ratio between the average multiplicities of π^0 -mesons in the MCE (CE) and in the GCE as closed circles (dashed line). Since pions are the lightest hadrons, we expect them to play the same role as massless particles in the analytical model. The ratios CE/GCE and MCE/GCE approach each other at energy of about 2 GeV and, thereafter, slowly converge to the GCE limit. For lower energies (less than ~ 2 GeV) the total particle multiplicity in the MCE is ~ 2 , as the dominant channels are $\pi^0 + \pi^0$ and $\pi^+ + \pi^-$. Therefore, while in the GCE the mean multiplicity of π^0 -mesons is proportional to the energy, in the MCE it is almost constant and $\sim 2/3$. This explains why the ratio $\langle \pi^0 \rangle_{\text{MCE}} / \langle \pi^0 \rangle_{\text{GCE}}$ has a nearly hyperbolic shape at small energies.

On the other hand, the increase of the ratio $\langle \pi^0 \rangle_{\text{CE}} / \langle \pi^0 \rangle_{\text{GCE}}$ as energy decreases is due to the higher temperature in the canonical ensemble for small systems with the same energy density (see Section II C and the discussion above). As the heavy (i.e. with $m \gg T$) neutral particle multiplicity is proportional to $\exp[-m/T]$ according to Eq. (17), the ratio $\langle n_h \rangle_{\text{CE}} / \langle n_h \rangle_{\text{GCE}}$ turns out to be proportional to:

$$\frac{\langle n_h \rangle_{\text{CE}}}{\langle n_h \rangle_{\text{GCE}}} \sim \exp \left[-m_h \left(\frac{1}{T(V, \varepsilon)} - \frac{1}{T(\infty, \varepsilon)} \right) \right] \quad (30)$$

which grows exponentially with the particle mass. Indeed, in Figs. 7-10 the ratios for heavy neutral particles η , ρ , f_2 and J/ψ show the expected behavior, with a “canonical” enhancement larger for heavier particles. For J/ψ , it is so large that it exceeds the “micro-canonical” enhancement, which, conversely, tends to diminish at larger masses. Indeed, for a sufficiently large mass, this enhancement disappears as it should have been expected from the result of the analytical model discussed in Section II.

In fact, as in the case of the analytical model, we observe a strong enhancement of the average multiplicities in the MCE with respect to the GCE near the production threshold. This effect is larger for lighter particles: about a factor ~ 10 in the ratio MCE/GCE for the second lightest neutral meson η and ~ 3.5 for J/ψ meson (see Figs. 7 and 10). These are remarkable figures, as the threshold peaks actually correspond to the “second” peaks observed in the analytical model (see Figs. 2 (left) and (right) as well as Fig. 3, (right)). The first peak (the largest one) is forbidden due to momentum conservation in the MCE that allows no less than two particles in the final state. A similar effect can be seen in the analytical model (Fig. 2)

for non-vanishing charge, because $Q = 2$ in the MCE also demands at least two particles in the final state. One can also see that the peaks in the full hadron gas have a gradual increase which is typical for the “second” peak (compare Figs. 2 (left) and (right)) in the analytical model.

In full hadron gas π^+ and π^- -mesons play the role of the massless charged particles in the analytical model considered in Section II. The “second peak” of neutral heavy particle multiplicities is present similar to that in Figs. 2 and 3. This “second peak” is stronger than in an analytical model, because the GCE energy density in the real hadron gas is a far larger for the same $T = 160$ MeV. This is due to much larger number of particle species. An exact momentum conservation makes the whole effect for the “second peak” even stronger.

The exact energy-momentum conservation is responsible for a non-smooth variation of the mean multiplicities as a function of the energy of the system. In fact, the opening of a new channel at a total energy E implies a rapid change of the behavior of this function. In order to show this, it is appropriate to have a closer look at the MCE/GCE ratios for π^0 , η , ρ -mesons. In Fig. 11 a zoom thereof is shown on the energy range 0.6-1.5 GeV, where threshold energies for relevant channels are marked. It is evident that there is a full correspondence between them and the slope changes. The general pattern can be understood from the analysis of the analytical massless-heavy particle model. The MCE/GCE ratio of massless particles gradually decreases to the threshold of heavy particle production. At the threshold, massless particle yield drops and heavy ones are enhanced. Likewise, in the full hadron gas π^0 -mesons play the role of the lightest particles and always show a drop whenever a new channel is opened. All other particles behave like “heavy” ones near their threshold, but when a new heavier particle threshold opens up, their MCE/GCE ratio drops, just as the lightest particles would do. For instance, in the middle panel of Fig. 11, the η production sets in at the $\eta\pi^0$ channel threshold and the curve raises until the energy $m_{\pi^0} + m_\rho$ is reached. Here the channel $\rho\pi^0$ is opened and, just a little later, $\omega\pi^0$ too. The production onset of these two new heavier particles downgrades η to the role of a light particle and a decrease in the slope of the ratio $\langle\eta\rangle_{MCE}/\langle\eta\rangle_{GCE}$ is implied. For little larger energy, the threshold for the $\eta\eta$ channel entails a new step up in the slope of the η ratio. Then, for $E > m_\eta + m_\rho$, the channels $\eta\rho$ and $\eta\omega$ are open and, consequently, both η and ρ ratios increase. Then, for energies larger than ~ 1.5 GeV, the number of allowed channels increases quickly and the ratio MCE/GCE shows a rocky pattern, which smoothes out thereafter.

Finally, we turn the quantities which might be, in general, related to the experimental results on hadron production in collisions of relativistic particles. Under simplifying assumptions that the hadron production is statistical and the system energy is proportional to the collision energy, hadronic final states can be identified with different configurations in the MCE. Consequently the dependence of the ratios of the mean hadron multiplicities in the MCE on the system energy may be expected to be related to the collision energy dependence of these ratios.

The dependence of $\langle\pi^0\rangle$ on the system energy calculated by use of the MCE and the GCE is presented in Fig. 12. There is a smooth and monotonic increase of $\langle\pi^0\rangle$ in the GCE. A very different behavior is seen in the MCE. For $E < 3$ GeV the pion yield reveals a non-monotonic, rocky dependence. In the very nearly of the threshold, $\langle\pi^0\rangle = 2$. This is due to the small difference between the mass of π^0 and π^\pm (~ 5 MeV) that prevent charged pions from being produced. For slightly higher energy, as the channel $\pi^+ + \pi^-$ is open too, $\langle\pi^0\rangle$ quickly decrease to $\sim 2/3$ as said before.

In Fig. 13, the ratios $\langle\eta\rangle/\langle\pi^0\rangle$, $\langle\rho\rangle/\langle\pi^0\rangle$ and $\langle J/\psi\rangle/\langle\pi^0\rangle$ are plotted as a function of the system energy. In the GCE the ratios are independent of the energy, whereas in the MCE a rocky pattern is predicted. Only above $E > 5$ GeV the ratios smoothly approach from above the GCE limit.

In this work we made the assumption of a constant energy-density, that is, in every plot the volume is proportional to the energy of the system. Different hypothesis (for instance an equal constant temperature for GCE and CE and an energy-density for MCE: $\varepsilon(T) = E/\langle V\rangle_{CE}$) would lead us to a completely different behavior of hadron multiplicities near the threshold with the same thermodynamical limit. A close comparison with the data could shed some light on which is the right critical quantity at hadronization.

IV. SUMMARY

We calculated and discussed the threshold behavior of particle multiplicities in relativistic gases. The micro-canonical formulations of two models were used. Firstly, analytical formulas were derived for the gas of massless and heavy particles. Second, the results for the hadron-resonance gas are obtained by use of the Monte-Carlo procedure. In both models the ratio of heavy to light particles near the threshold is enhanced in comparison to one in the grand

canonical limit. At low system energies ($E < 3$ GeV) a non-monotonic and rocky dependence of hadron multiplicities and their ratios on the system energy is obtained within hadron-resonance gas model.

The problem whether these unexpected micro-canonical threshold effects can be seen in the experimental data on collisions of relativistic particles is left for a future study.

Acknowledgments

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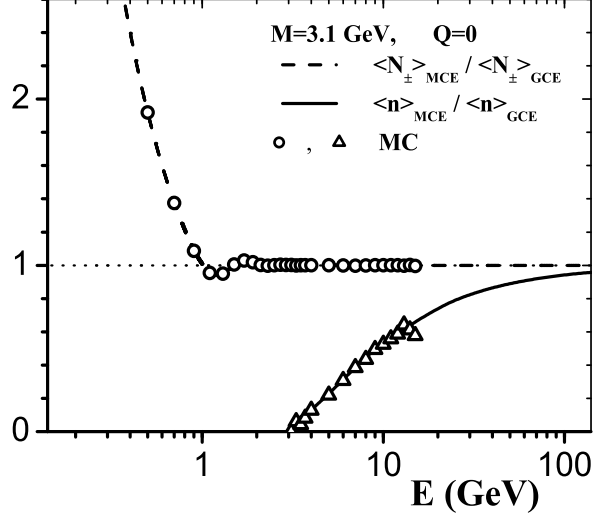


FIG. 1: The dependence of the ratios of the mean particle multiplicities calculated in the MCE to those the GCE on the total energy of the system. The gas with zero net charge $Q = 0$ of massless charged particles ($g_{\pm} = 1$) and heavy neutral particles ($M = 3.1 \text{ GeV}$ and $G = 1$) is considered. The ratios for massless particles, $\langle N_{\pm} \rangle_{MCE} / \langle N_{\pm} \rangle_{GCE}$, (dashed line) and heavy neutral particles, $\langle n \rangle_{MCE} / \langle n \rangle_{GCE}$, (solid line) are plotted. The corresponding Monte-Carlo (MC) calculations are shown by circles and triangles. The calculations are performed for $T = 160 \text{ MeV}$ and $E \equiv \langle E \rangle_{GCE}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE.

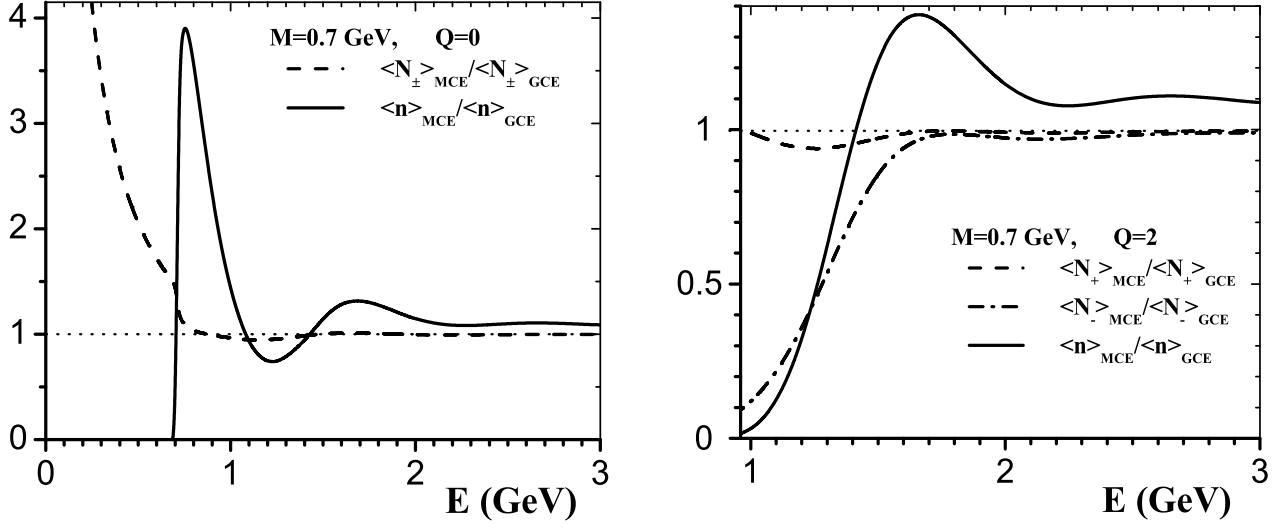


FIG. 2: Left: The same as in Fig. 1 but for heavy particle mass $M = 0.7$ GeV.

Right: The same as in Fig. 1 but for heavy particle mass $M = 0.7$ GeV and the system net charge $Q = 2$.

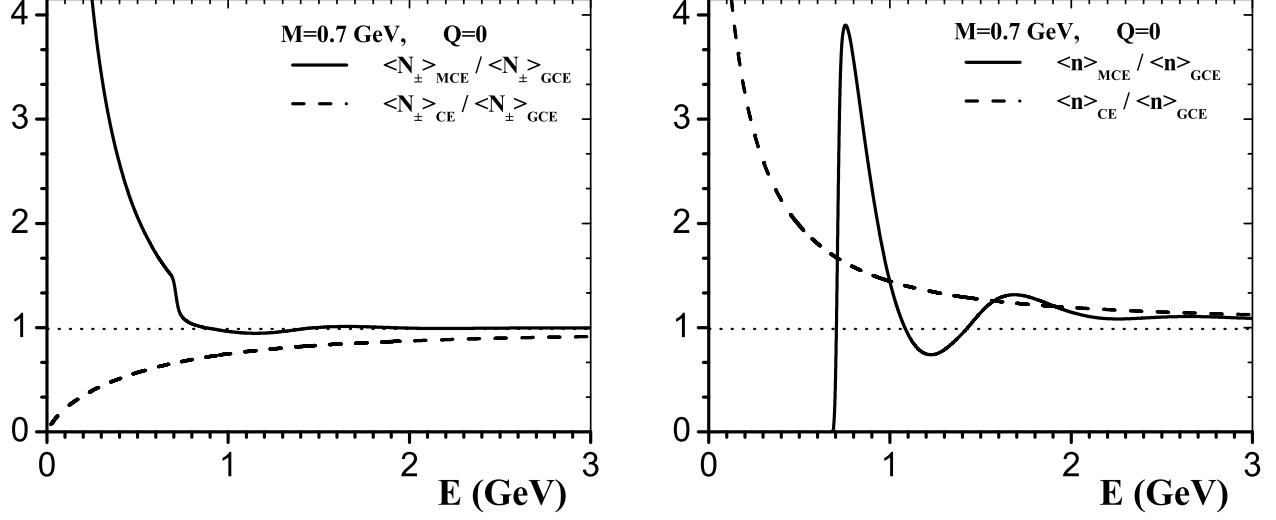


FIG. 3: The dependence of the ratios of the mean particle multiplicities calculated in the MCE and the CE to those in the GCE on the total energy of the system. The gas with zero net charge $Q = 0$ of massless charged particles ($g_{\pm} = 1$) and heavy neutral particles ($M = 0.7$ GeV and $G = 1$) is considered. The ratios for massless particles, $\langle N_{\pm} \rangle_{MCE} / \langle N_{\pm} \rangle_{GCE}$ (solid line) and $\langle N_{\pm} \rangle_{CE} / \langle N_{\pm} \rangle_{GCE}$ (dashed line) are shown in the left plot whereas the ratios for heavy particles, $\langle n_{\pm} \rangle_{MCE} / \langle n_{\pm} \rangle_{GCE}$ (solid line) and $\langle n_{\pm} \rangle_{CE} / \langle n_{\pm} \rangle_{GCE}$ (dashed line) are shown in the right plot. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE and in the CE.

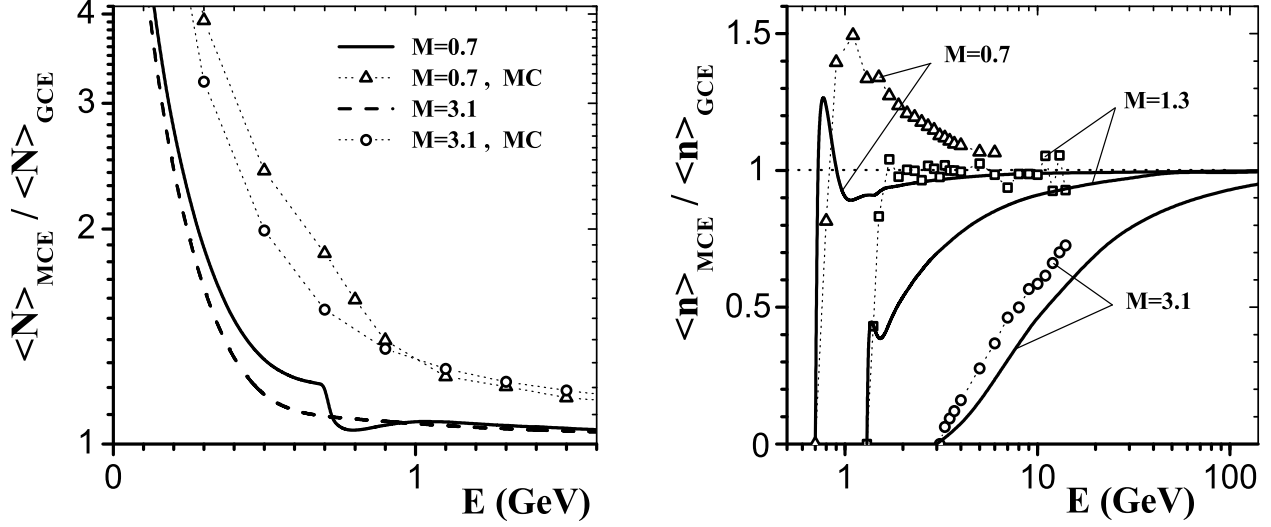


FIG. 4: The dependence of the ratios of the mean particle multiplicities calculated in the MCE to those in the GCE on the total energy of the system. The gas of massless neutral particles ($g_{\pm} = 1$) and heavy neutral particles ($M = 0.7, 1.3, 3.1$ GeV and $G = 1$) is considered. Left: The ratios for massless particles, $\langle N_{\pm} \rangle_{MCE} / \langle N_{\pm} \rangle_{GCE}$ calculated within the analytical model without momentum conservation are shown by solid ($M = 0.7$ GeV) and dashed ($M = 3.1$ GeV) lines. The corresponding Monte-Carlo results with momentum conservation are indicated by triangles and circles, respectively. Right: The ratios for heavy particles, $\langle n_{\pm} \rangle_{MCE} / \langle n_{\pm} \rangle_{GCE}$ calculated within the analytical model without momentum conservation are shown by solid lines. The corresponding Monte-Carlo results with momentum conservation are indicated by triangles, squares and circles. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE. Connecting lines between the MC-dots are drawn to guide the eyes.

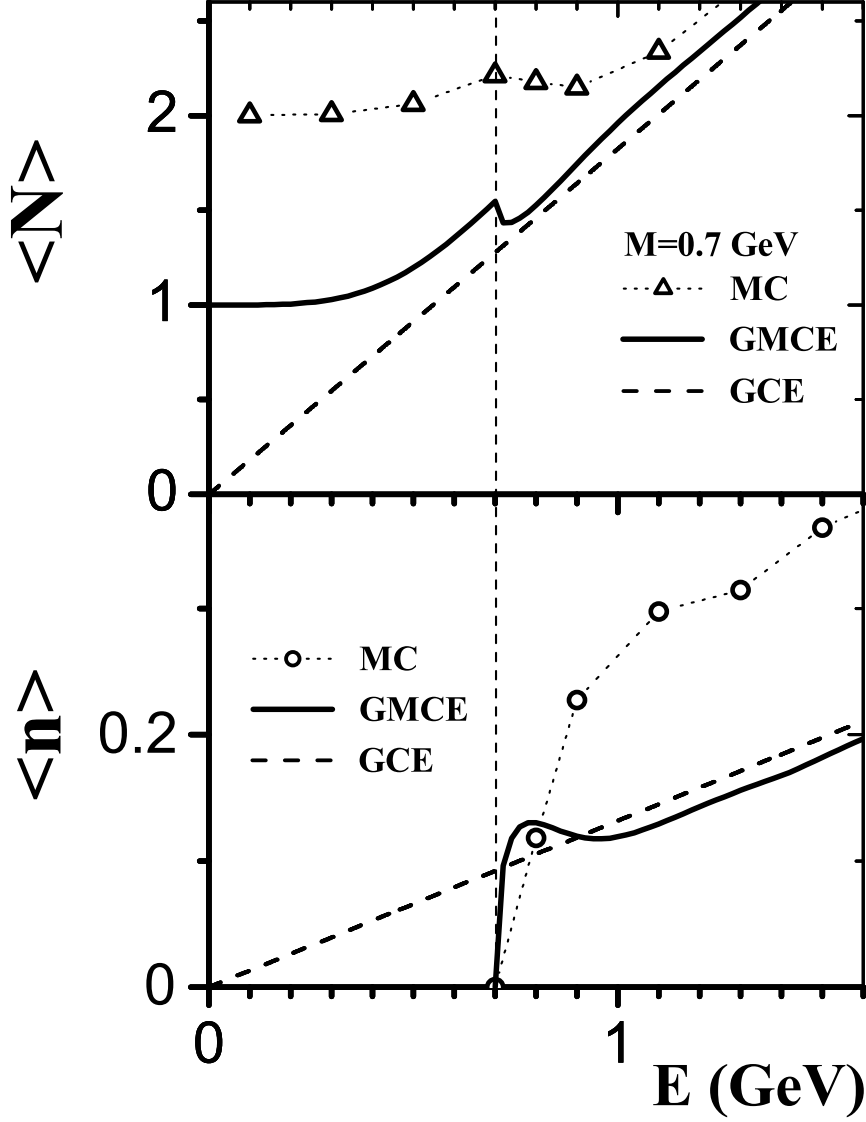


FIG. 5: The dependence of the mean particle multiplicities calculated in the MCE, the GMCE and the GCE on the total energy of the system. The gas of massless neutral particles ($g_{\pm} = 1$) and heavy neutral particles ($M = 0.7$ GeV and $G = 1$) is considered. Top panel shows the multiplicity of massless particles, $\langle N_{\pm} \rangle$, calculated in the GCE (dashed line), in the GMCE (solid line) and in the MCE with momentum conservation (triangles). Bottom panel shows the corresponding results for the heavy particles. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE. Connecting lines between the MC-dots are drawn to guide the eyes.

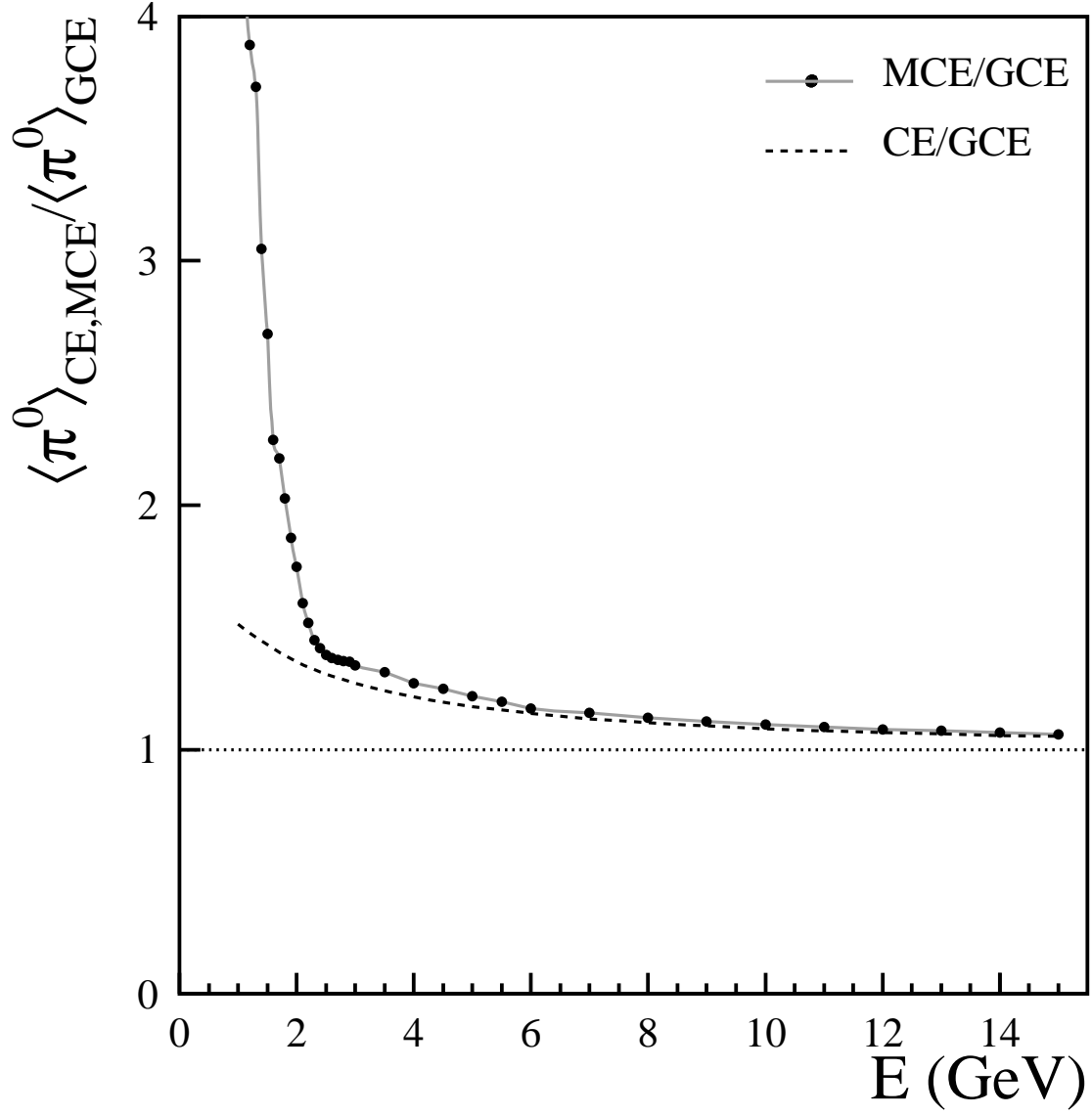


FIG. 6: The energy dependence of the ratio of the mean π^0 -meson multiplicity in the MCE and the CE to those in the GCE obtained within the hadron-resonance gas model. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{\text{GCE}}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE and the CE. Connecting lines between the MC-dots are drawn to guide the eyes.

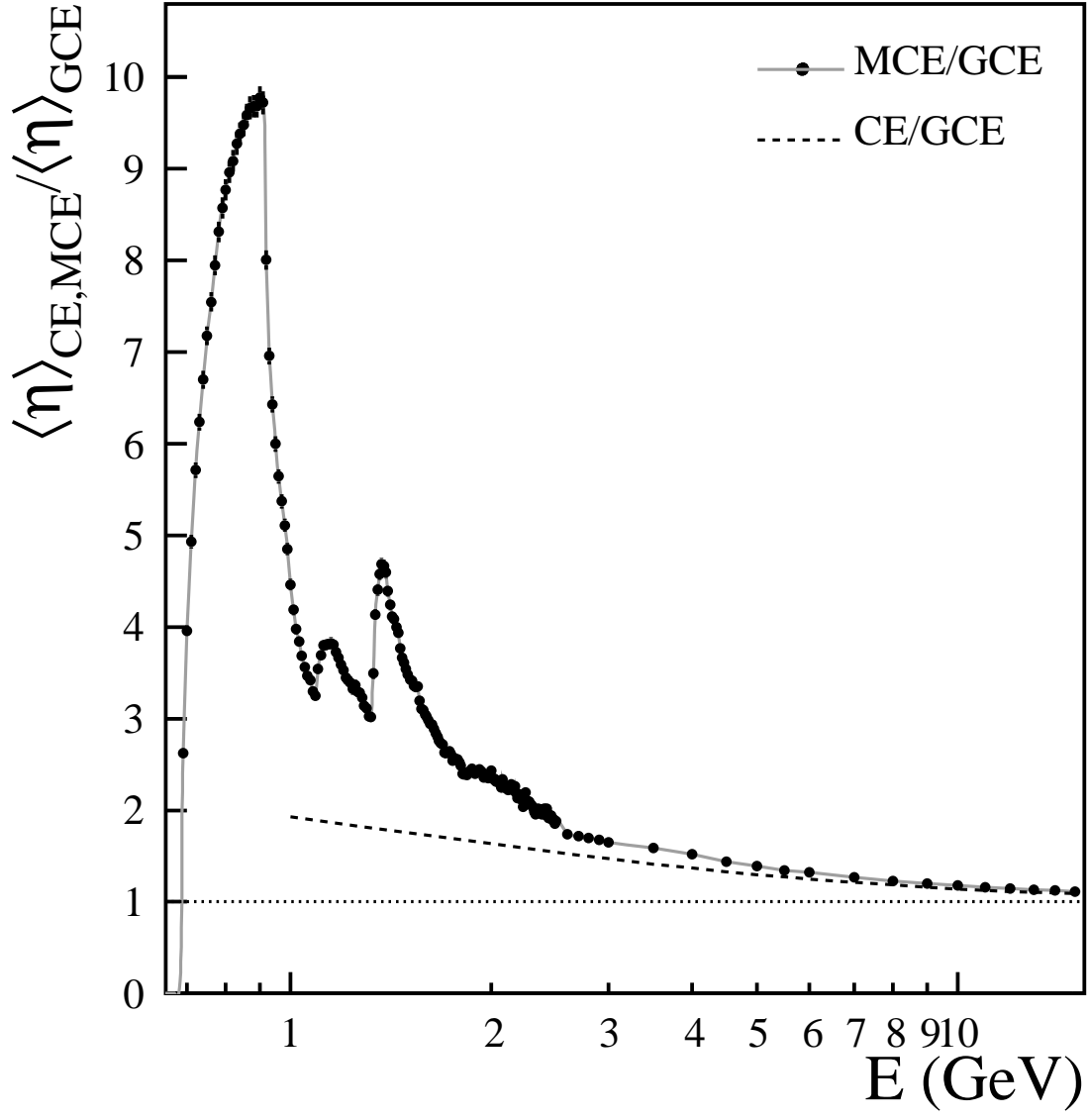


FIG. 7: The same as in Fig. 6 but for η meson.

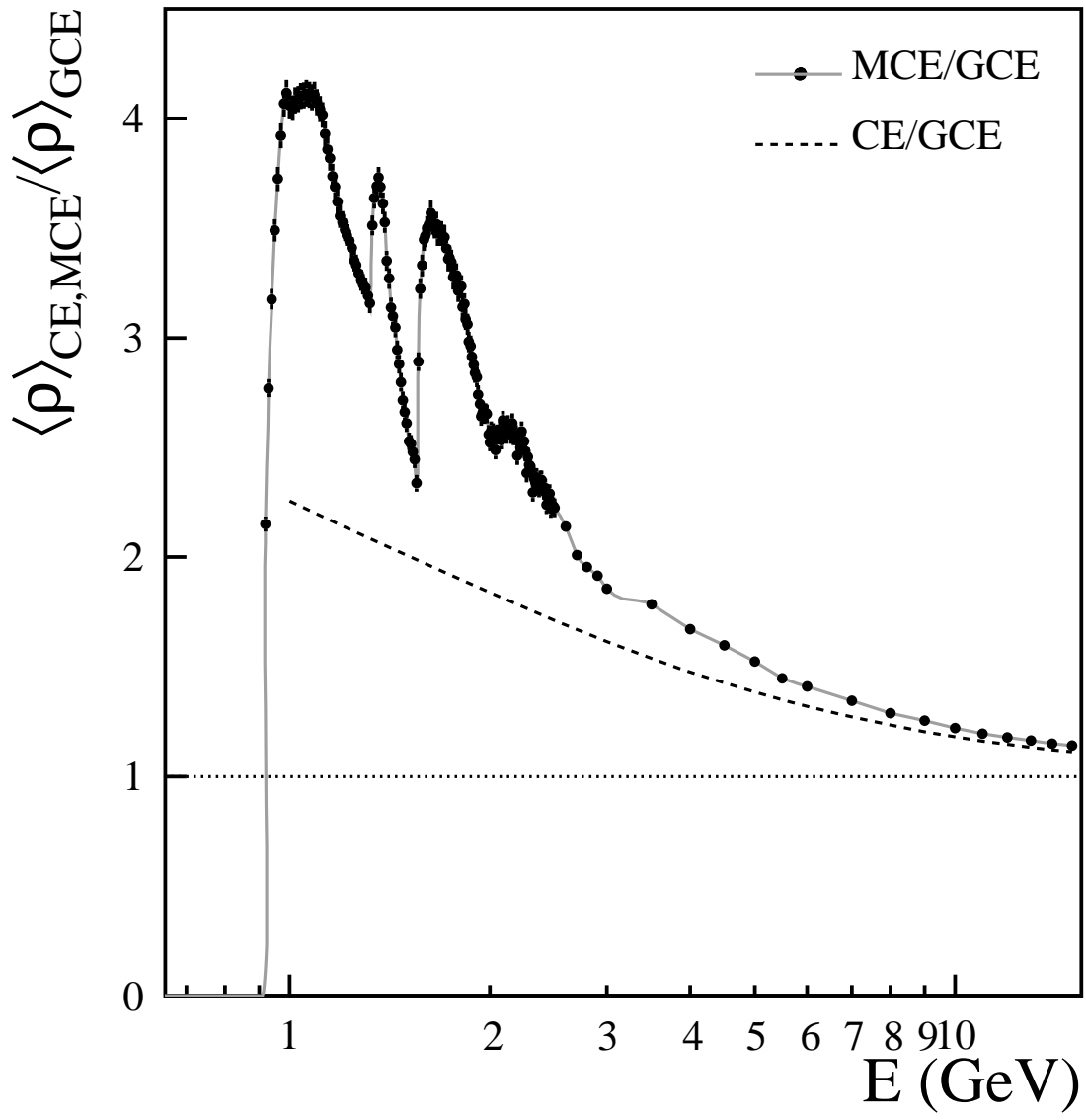


FIG. 8: The same as in Fig. 6 but for ρ meson.

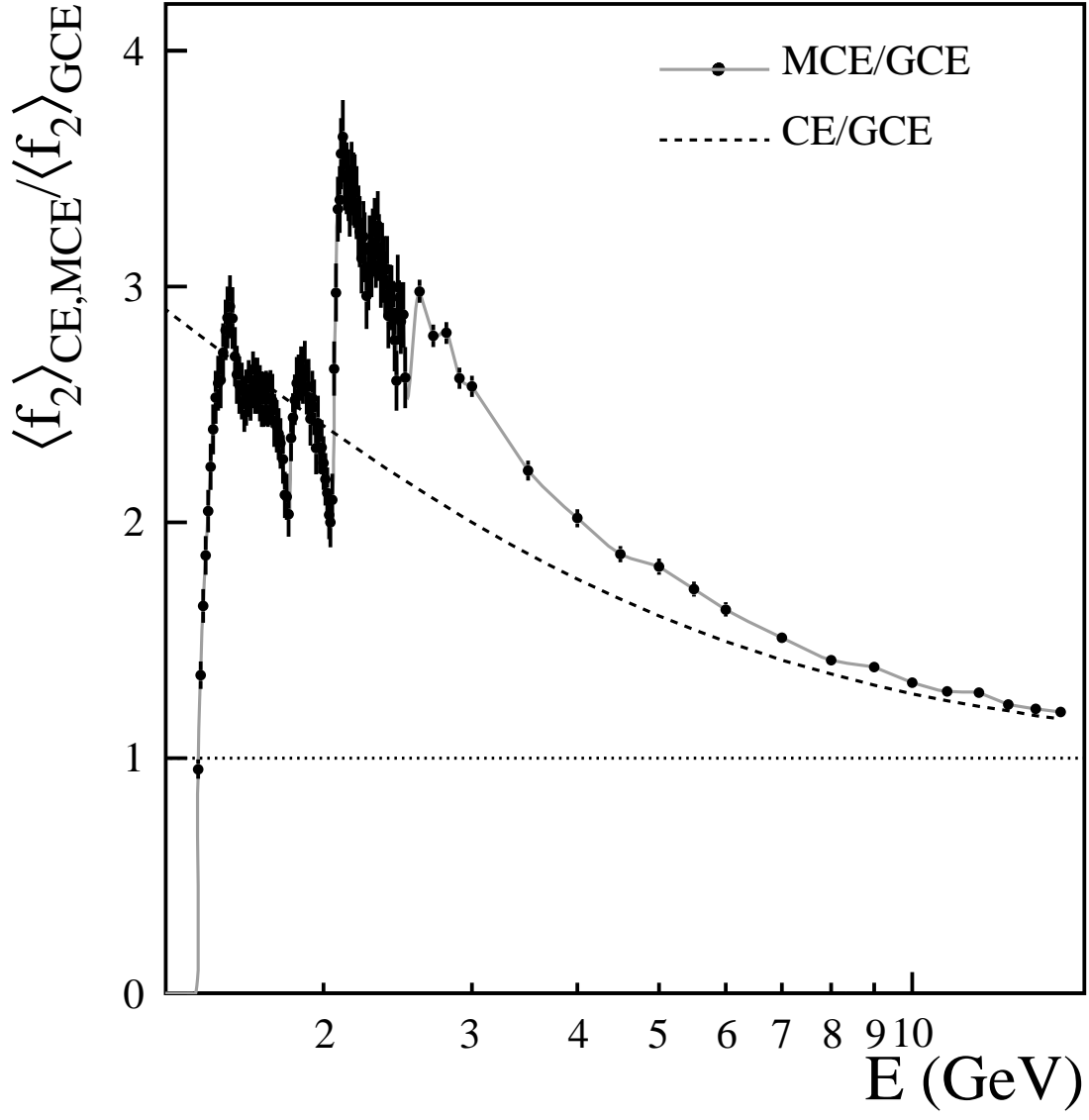


FIG. 9: The same as in Fig. 6 but for f_2 meson.

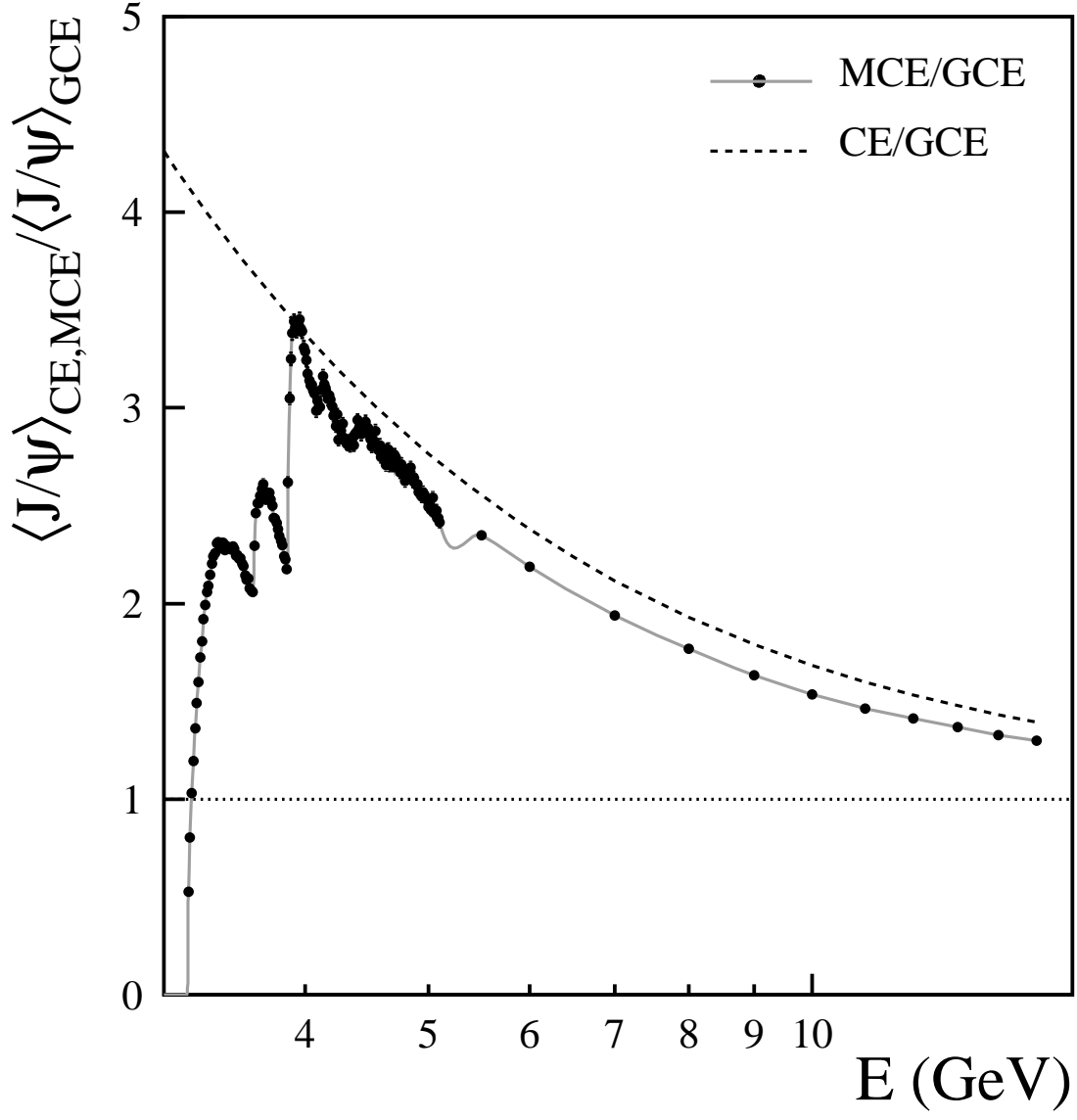


FIG. 10: The same as in Fig. 6 but for J/ψ meson.

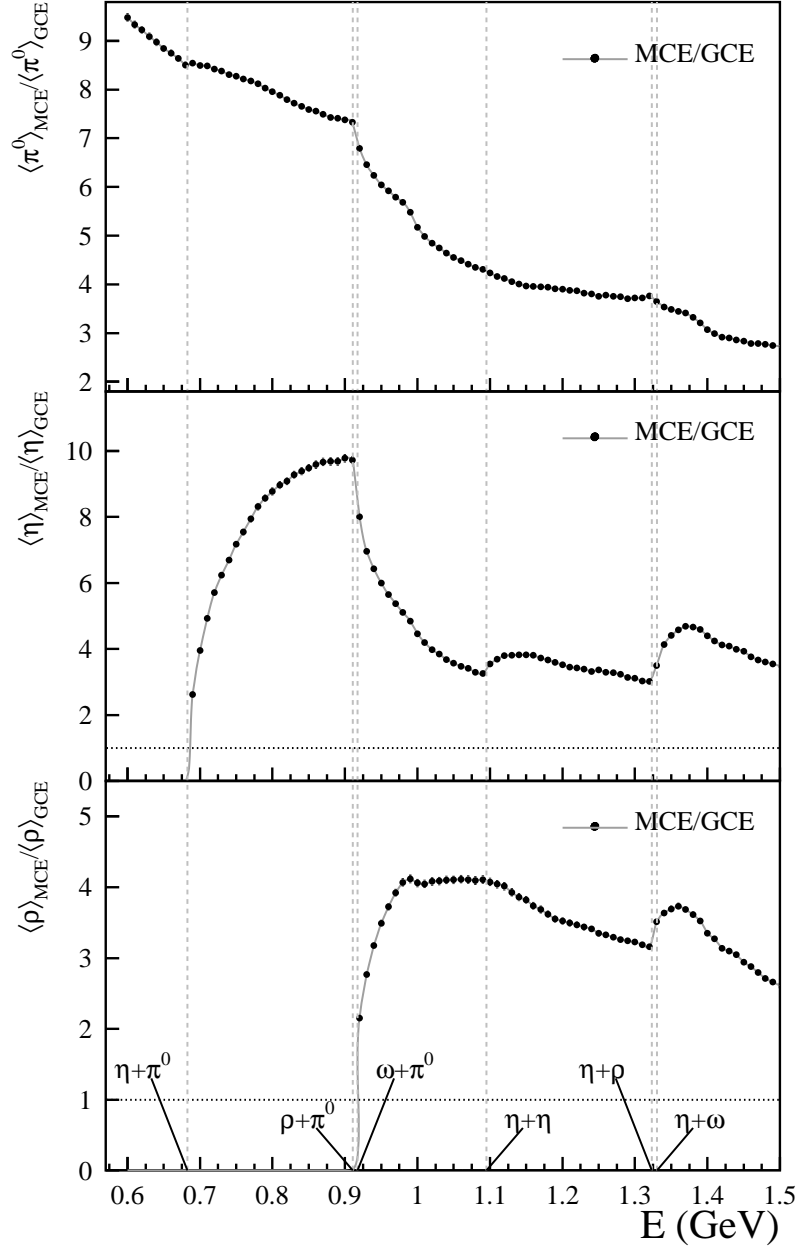


FIG. 11: The energy dependence of the ratio of the mean multiplicities of π^0 (top), η (middle) and ρ (bottom) mesons in the MCE and the CE to those in the GCE obtained within the hadron-resonance gas model. The vertical dashed lines indicates the threshold energies of several two particle channels which are responsible for the observed structures. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in the GCE and for $E \equiv \varepsilon(T)V$ in the MCE. Connecting lines between the MC-dots are drawn to guide the eyes.

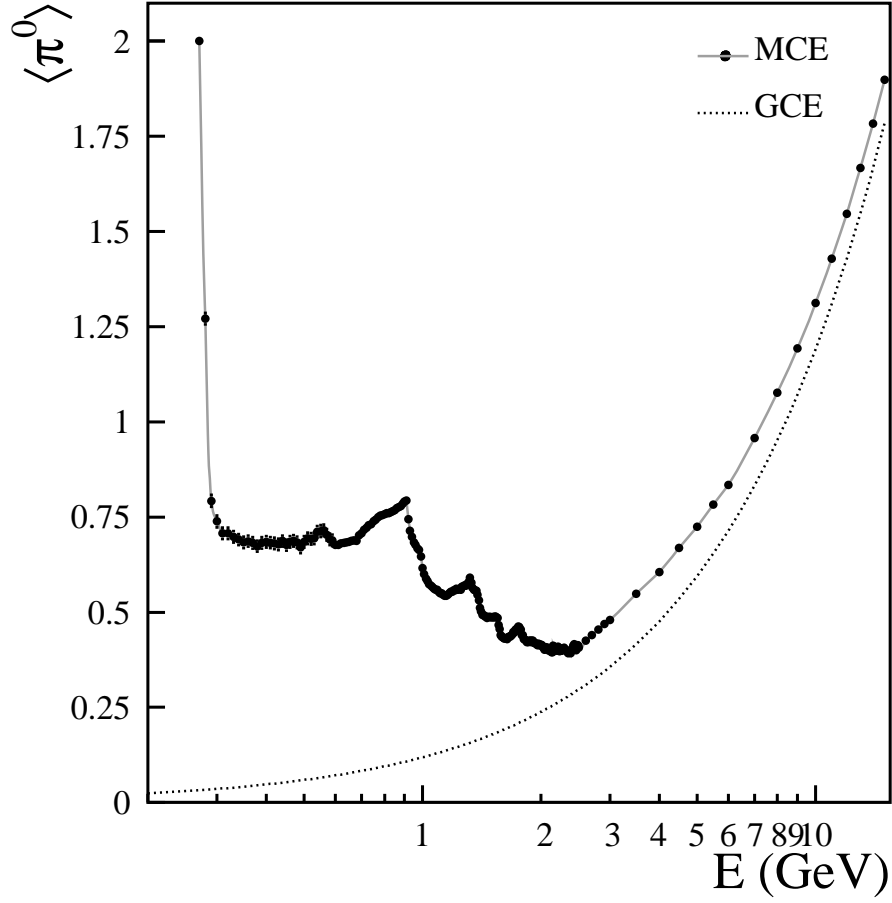


FIG. 12: The dependence of the mean multiplicity of π^0 mesons on the system energy obtained within the hadron-resonance gas model by use of the micro-canonical (solid line) and grand canonical (dotted line) ensembles. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in GCE and for $E \equiv \varepsilon(T)V$ in MCE. Connecting lines between the MC-dots are drawn to guide the eyes.

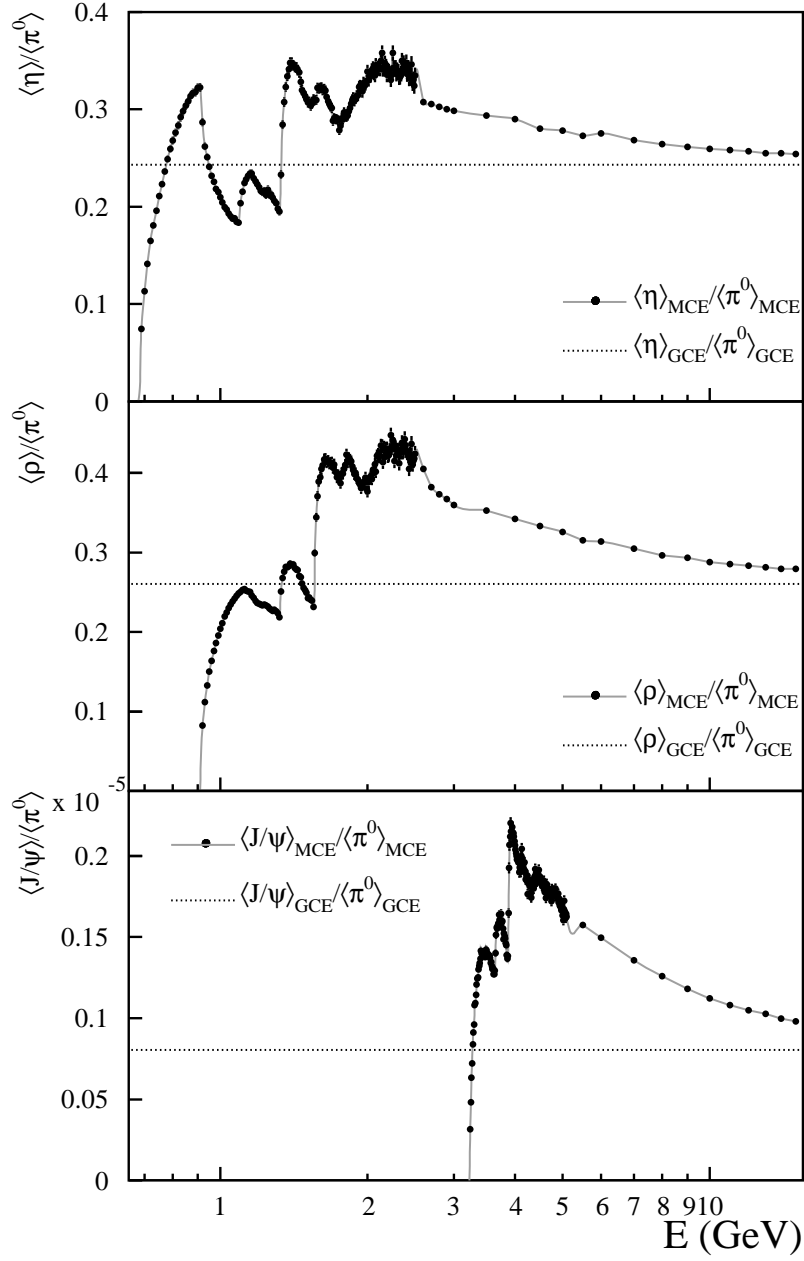


FIG. 13: The energy dependence of the ratios $\langle \eta \rangle / \langle \pi^0 \rangle$, $\langle \rho \rangle / \langle \pi^0 \rangle$ and $\langle J/\psi \rangle / \langle \pi^0 \rangle$ obtained within the hadron-resonance gas model by use of the micro-canonical (points and solid line) and grand canonical (dotted line) ensembles. The calculations are performed for $T = 160$ MeV and $E \equiv \langle E \rangle_{GCE}$ in GCE and for $E \equiv \varepsilon(T)V$ in MCE. Connecting lines between the MC-dots are drawn to guide the eyes.